SHADOW SYSTEMS OF CONVEX SETS

BY

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ABSTRACT

An s-system of convex sets is the system of shadows of a given convex set cast on to a subspace by a beam of light whose direction varies. Here the convexity properties of s-systems are investigated, and, in the final section, a relationship with the projection functions of convex sets is established.

In three-dimensional space, the shadow of a convex set cast on to a plane by a parallel beam of light is a convex region. If we let the direction of the beam vary, we get a system of convex regions in the plane which will be called a *shadow system of convex sets*, or, more briefly, an *s-system*. The purpose of this short paper is to investigate the properties of s-systems. In particular it will be shown that if an s-system is parametrised in a suitable way, many geometrical functionals such as the volume, surface area, diameter, etc., are convex functions of the system parameter.

Although there is no exact theory of duality in the study of convex bodies, s-systems seem, in some sense, to play the part of duals to Minkowski concave systems. This duality arises because, whereas an s-system consists of the projections of some convex set in higher-dimensional space, a Minkowski concave system may be considered as arising from the parallel sections of a such a set [2, p. 33].

S-systems are closely related to the process of Steiner symmetrisation [2, p. 69], and, in a one-dimensional form, occur implicitly in the works of many authors (compare, for example, the *continuous symmetrisation* of Pólya and Szegö [5, p. 200] and the *linear parameter system* of Rogers and Shephard [6, p. 95]).

§1. **Definitions and Elementary Properties.** In Euclidean space of $n+1$ dimensions, E^{n+1}, let ξ be any non-zero vector, K be any closed bounded convex set, and $\mathscr{H}$ any hyperplane (subspace of n dimensions). Then we define $S(\xi, K, \mathscr{H})$ (*the shadow of K on $\mathscr{H}$ in the direction ξ*) to be

$$\mathscr{H} \cap Z(K, \xi)$$

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where $Z(K, \xi)$ is the cylinder $\{x + t\xi \mid x \in K, -\infty < t < \infty\}$ containing K with generators in the direction ξ.

Let a be any fixed vector in E^{n+1} not parallel to $\mathscr{H}$. Since the definition is affine invariant there will be no loss of generality in assuming that a is a unit vector normal to $\mathscr{H}$. Let u be a variable vector parallel to $\mathscr{H}$, and let

$$K(u) = S(a + u, K, \mathscr{H}).$$

Then the system of convex sets $\{K(u)\}$, as u varies, is called an *s-system*, and u is the *system parameter*. The s-system will be said to *originate* from the set $K \subset E^{n+1}$.

Since, clearly, $S(a + u, x + \zeta a, \mathscr{H}) = x - \zeta u$ for any real number ζ, an alternative definition of $K(u)$ is

$$K(u) = \{x - \zeta u \mid x + \zeta a \in K\}$$

for each u. Written in this way, it is easy to see that if u is restricted to lie on a line, we obtain the linear parameter system of [6].

Since $S(\xi, K, \mathscr{H})$ is an affine image of the orthogonal projection of K on to a hyperplane normal to ξ, many elementary properties of s-systems follow immediately from the corresponding properties of orthogonal projections. For example: *If all the sets $K(u)$ of an s-system are centrally symmetric, then so is the set K from which they originate.* The corresponding result for orthogonal projections was first proved by Blaschke and Hessenberg, see [2, p. 124].

As a second example, we mention the (rather surprising) fact *if $\{K_1(u)\}$ and $\{K_2(u)\}$ are two s-systems such that*

$$v_n(K_1(u)) < v_n(K_2(u)) \tag{2}$$

for all u, then it is possible for

$$v_{n+1}(K_1) > v_{n+1}(K_2). \tag{3}$$

(Here $v_n(X)$ means the n-dimensional volume or content of the set X.) The corresponding result for orthogonal projections must have been known for a long time, but does not seem to appear in the literature; we therefore give an example below. The question of whether (2) implies $v_{n+1}(K_1) < v_{n+1}(K_2)$ if we restrict K_1 and K_2 to be centrally symmetric is still open. An answer would be interesting since this question is dual (in some sense) to an unsolved problem of Busemann and Petty [3, p. 88] about the cross-sections of centrally symmetric bodies.

Let K_1 be any ball in E^3, and K_2^* be any non-spherical body of constant brightness ([2, p. 140] and [1, p. 151]) whose orthogonal projection (and therefore shadow) in any direction is equal in volume to that of K_1.

By Cauchy's surface area formula [2, p. 48] and the isoperimetric theorem [2, p. 111] it follows that $v_{n+1}(K_1) > v_{n+1}(K_2^*)$. If we dilate K_2^* slightly, we obtain a body K_2 which satisfies (2), and, if the dilation is small enough, will satisfy (3) also.

Let H_0, H_1 be two given convex sets in $\mathscr{H}$. Then it will not, in general, be possible to find an s-system which contains them both. A criterion for this is:

(4) *Let H_0, H_1 be any two closed, bounded convex sets and u_0, u_1 be any two vectors in E^n. Then there exists an s-system $\{K(u)\}$ such that*

$$K(u_0) = H_0, \; K(u_1) = H_1$$

if any only if $P(H_0, \mathscr{R}) = P(H_1, \mathscr{R})$, where $P(H, \mathscr{R})$ is the orthogonal projection of H on to a hyperplane $\mathscr{R} \subset \mathscr{H}$ normal to the vector $u_0 - u_1$.

The corresponding criterion for more than two sets H_i is not known. Clearly (4) is necessary since, by the properties of orthogonal projection, both $P(K(u_0), \mathscr{R})$ and $P(K(u_1), \mathscr{R})$ are equal to $P(K, \mathscr{R})$. It is also sufficient, for if a is any unit vector in E^{n+1} normal to the hyperplane $\mathscr{H}$ in which H_0, H_1 lie, then we may put

$$K = Z(H_0, a + u_0) \cap Z(H_1, a + u_1)$$

and it is easily verified that $H_0 = K(u_0)$ and $H_1 = K(u_1)$. This proves (4).

There will be many s-systems containing the given sets H_0 and H_1, but for any such system $\{K'(u)\}$, it is clear that $K' \subset K$. Hence the system defined above is, in an obvious sense, the 'maximal' one. It may be called a *linear s-system* by analogy with a linear Minkowski system, which is the 'minimal' concave system containing two given sets.

If $H_0 = K(u_0)$, $H_1 = K(u_1)$, then the system of sets $K((1-\theta)u_0 + \theta u_1)(0 \leqq \theta \leqq 1)$, may, besides being part of an s-system, be a Minkowski linear system. By [2, p. 94] this occurs if and only if one of the sets H_0, H_1 can be produced from the other by being 'stretched' in a direction normal to the hyperplane $\mathscr{R}$.

If H_1 is the reflection of H_0 in the hyperplane $\mathscr{R}$, then H_0 and H_1 satisfy condition (4) and we may, as above, construct a linear s-system $\{K(u)\}$ with $K(u_0) = H_0$, $K(u_1) = H_1$ for suitable vectors u_0, u_1. Such a system is invariant under reflection in $\mathscr{R}$. Further $K(\frac{1}{2}(u_0 + u_1))$ is the result of applying Steiner symmetrisation [2, p. 69] to H_0 (or to H_1). This is the relation between s-systems and Steiner symmetrisation mentioned in the introduction.

§2. Convexity Properties of s-systems.

(5) *Let $\{K(u)\}$ be an s-system of convex sets in E^n. Then the volume $v_n(K(u))$ is a convex function of the system parameter u.*

We present two short proofs of this basic result:

(i) As we have already noted, if u is restricted to lie on a line, then $\{K(u)\}$ is a linear parameter system, and so by [6, p. 95], $v_n(K(u))$ is a convex function of u. Since this is true for every line $v_n(K(u))$ is a convex function of u in E^n.

(ii) Since a is a unit vector normal to $\mathscr{H}$, the equality

$$v_n(S(a+u, K, \mathscr{H})) = (n+1)v(a+u, K, K, \cdots, K)$$

holds, where the expression on the right is the mixed volume of the line segment corresponding to the vector $a+u$, and the set K taken n times. But Minkowski proved that this mixed volume is a convex function of $a+u$ (see [2, p. 44]) and so $v_n(K(u))$ is a convex function of u.

Minkowski's argumenta lso applies to the mixed volume $v(a+u, K_1, K_2, \cdots, K_n)$, and so we deduce the following generalisation of (5):

(6) *Let* $\{K_1(u)\}, \{K_2(u)\}, \cdots, \{K_n(u)\}$ *be any n, not necessarily distinct, s-systems with the same parameter u. Then the mixed volume* $v(K_1(u), K_2(u), \cdots, K_n(u))$ *is a convex function of u.*

A number of special cases are of interest. Let $K_1 = K_2 = \cdots = K_r = K$ and let $K_{r+1} = K_{r+2} = \cdots = K_n = B^n$, an n-dimensional unit ball lying in $\mathscr{H}$, so that $B^n(u) = B^n$ for all u. Taking $r = n-1$ we deduce:

(7) *If* $\{K(u)\}$ *is any s-system in* E^n, *then the* $(n-1)$*-dimensional surface area* $a(K(u))$ *is a convex function of u.*

An alternative proof of (7) can be constructed from (10) and the fact that, by Cauchy's surface area formula, the surface area is a constant multiple of the average area of projection on to a hyperplane. If we take $r = 1$ we obtain:

(8) *If* $\{K(u)\}$ *is any s-system, then the mean width of* $K(u)$ *is a convex function of u.*

Other values of r lead to convex functionals, of which the following special case will be used later:

(9) *Let* $\{K(u)\}$ *be an s-system in* E^n *with the property that all the sets* $K(u)$ *have dimension r. Then* $v_r(K(u))$ *is a convex function of u.*

A system of this type arises (for $r < n$) if and only if K is r-dimensiona l.

Futher special cases of (6) arise when we take some of the sets K to be balls of dimension lower than n. Let $\mathscr{R}$ be any r-dimensional subspace of $\mathscr{H}$, and, as above, write $P(H, \mathscr{R})$ for the orthogonal projection of any set H onto $\mathscr{R}$. Let $K_{r+1} = K_{r+2} = \cdots = K_n = B^{-r}$ be an $(n-r)$-dimensional ball in $\mathscr{H}$ normal to $\mathscr{R}$. Then $B^{n-r}(u) = B^{n-r}$ (for all u) and since the mixed volume

$$v(K(u), \cdots, K(u), B^{n-r}, \cdots, B^{n-r})$$

(in which $K(u)$ occurs r times and B^{n-r} occurs $(n-r)$ times) is a constant multiple of $v_r(P(K(u), \mathscr{R}))$, we deduce:

(10) *Let* $\{K(u)\}$ *be any s-system in* E^n *and* $\mathscr{R}$ *any linear subspace of r dimensions. Then* $v_r(P(K(u), \mathscr{R}))$ *is a convex function of u.*

The case $r = n-1$ (so that $\mathscr{R}$ is a hyperplane in E^n) leads to the alternative proof of (7) mentioned above. Since the supremum of a set of convex functions is also a convex function, we may deduce also:

(11) *Let $\{K(u)\}$ be any s-system in E^n. Then the maximum brightness of $K(u)$ is a convex function of u.*

The *maximum brightness* (defined by analogy with 'sets of constant brightness') is the maximum $(n-1)$-dimensional volume of the projections of $K(u)$ on to hyperplanes.

The case $r = 1$ yields:

(12) *Let $\{K(u)\}$ be any s-system. Then the width of $K(u)$ in a given fixed direction is a convex function of u.*

If we average over all directions, we obtain an alternative proof of (8). If we take the supremum over all directions, we obtain:

(13) *Let $\{K(u)\}$ be any s-system, then the diameter of of $K(u)$ is a convex function of u.*

Statement (10) may also be established from:

(14) *If $\{K(u)\}$ is any s-system, and $\mathscr{R}$ is any r-dimensional subspace, then $\{P(K(u),\mathscr{R})\}$ is also an s-system, the system parameter being $P(u, \mathscr{R})$.*

For let $\mathscr{R}^*$ be any $(r+1)$-dimensional subspace through $\mathscr{R}$ and normal to $\mathscr{H}$ Then projecting orthogonally on to $\mathscr{R}^*$ we see that

$$\begin{aligned} P(K(u),\mathscr{R}) &= P(K(u),\mathscr{R}^*) = P(S(a+u,K,\mathscr{H}),\mathscr{R}^*) \\ &= S(P(a+u,\mathscr{R}^*),P(K,\mathscr{R}^*),P(\mathscr{H},\mathscr{R}^*)) \\ &= S(a+P(u,\mathscr{R}^*),P(K,\mathscr{R}^*),P(\mathscr{H},\mathscr{R}^*)) \end{aligned}$$

which, as u varies, is an s-system in $\mathscr{R}$ with parameter $P(u,\mathscr{R}^*) = P(u,\mathscr{R})$.

Other convex functionals can be defined in terms of convex polytopes inscribed in $K(u)$:

(15) *Let $f_s(X)$ be the functional defined as the maximum n-dimensional volume of all polytopes with at most s vertices included in the set X. Then for any s-system $\{K(u)\}$ in E^n, $f_s(K(u))$ $(s \geqq n+1)$ is a convex function of u.*

Let Π_s be any convex polytope, with at most s vertices, included in the set K. Then $S(a+u,\Pi_s,\mathscr{H}) \subset S(a+u,K,\mathscr{H})$ and, further $S(a+u,\Pi_s,\mathscr{H})$ is a convex polytope with at most s vertices. Conversely, any polytope with at most s vertices included in $K(u)$ can arise in this way as a shadow of a suitable $\Pi_s \subset K$. Hence

$$f_s(K(u)) = \sup_{\Pi_s \subset K} v_n(S(a+u,\Pi_s,\mathscr{H})).$$

But, by (5), $v_n(S(a+u,\Pi_s,\mathscr{H})$ is a convex function of u for each Π_s, and so, being a supremum of convex functions, $f_s(K(u))$ is also a convex function of u. This proves (15).

In an exactly similar manner, using (6) instead of (5), we may deduce a statement corresponding to (15) concerning the maximal $(n-1)$-dimensional surface area of polytopes, with at most s vertices, included in $K(u)$.

(16) *Let $j_s(X)$ be the maximum sum of the lengths of the $\frac{1}{2}s(s-1)$ line segments joining s points belonging to the convex set X. Then for any shadow system $\{K(u)\}$, $j_s(K(u))$ is a convex function of u.*

For the proof, take T_s as any set of s points belonging to K. Then $S(a+u, T_s, \mathscr{H})$ is a set of s points belonging to $K(u)$, and the sum of the lengths of the joins of these points is

$$j(T_s, u) = \sum_{1 \leqq i \leqq j \leqq s} v_1(S(a+u, (t_i t_j), \mathscr{H}))$$

where $T_s = \{t_1, \cdots, t_s\}$ and $(t_i t_j)$ is the line segment joining t_i to t_j. By (9), with $r = 1$, each term $v_1(S(a+u, (t_i t_j), \mathscr{H})$ is a convex function of u, and so, being a sum of convex functions, $j(T_s, u)$ is also convex. Now

$$j_s(K(u)) = \sup_{T_s \subset K} j(T_s, u)$$

and so, being a supremum of convex functions, $j_s(K(u))$ is also a convex function of u. This proves (16).

If we put $s = 2$ in (16) we obtain (13) again.

Because of the relation between s-systems and Steiner symmetrisation mentioned at the end of §1, it follows that:

(17) *Let $f(X)$ be any functional defined on convex sets X in E^n, with the properties*:

(i) *for any s-system $\{K(u)\}$, $f(K(u))$ is a convex function of u, and*
(ii) *$f(X) = f(X')$, where X' is the reflection of X in any hyperplane,*
then the functional $f(X)$ is not increased by Steiner symmetrisation of X.

Thus, for example, the surface area, mean width, maximum brightness and diameter of a set tend to be decreased by Steiner symmetrisation (see (7), (8), (11), (13)). The volume $v_n(X)$ is left invariant by Steiner symmetrisation, which is a consequence of the fact that if $K(u)$ is any linear s-system joining $H_0 = K(u_0)$ and $H_1 = K(u_1)$, then $v_n(K(\lambda u_0 + (1-\lambda)u_1))$ is, for $0 \leqq \lambda \leqq 1$, a linear function of λ. Hence, for a symmetrical linear s-system, it is constant.

The converse of (16) is not true, in fact many interesting and important functionals, such as the circumradius, reciprocal of the inradius, reciprocal of the breadth, moment of inertia, electrical capacity, are decreased by Steiner symmetrisation, yet none of these are convex functions on general s-systems, or even on linear ones. The proofs of these asertions are omitted. For those concerning moments of inertia and electrical capacity (as well as other quantities of a physical nature), the reader ir referred to the work of Pólya and Szegö [5].

§3. **A Generalisation.** In the first two sections we considered s-systems $\{K(u)\}$ with one system parameter u. We now present a brief account of a generalisation which provides geometrical insight into the properties of the projection function $\hat{P}(K,R)(1 \leqq r \leqq n-1)$ discussed in recent publications of H. Busemann, G. Ewald and the author [4]. For notations and terminology, the reader is referred to this paper.

Let K be any closed bounded convex set in E^{n+}, $\mathscr{H}$ any n-dimensional linear subspace and $a_1, \cdots, a_t$ be unit vectors normal to $\mathscr{H}$ and to each other. Put

$$K(u_1, \cdots, u_t) = \{x - \sum_{i=1}^{t} \zeta_i u_i \mid x + \sum_{i=1}^{t} \zeta_i a_i \in K\} \tag{18}$$

for any set $\{u_1, \cdots, u_t\}$ of vectors parallel to $\mathscr{H}$. Then $\{K(u_1, \cdots, u_t)\}$ will be called, as the u_i vary, an *s-system of convex sets with t system parameters* $u_1, \cdots, u_t$. If one system parameter varies, and the remainder are fixed, $\{K(u_1, \cdots, u_t)\}$ is an s-system as previously defined. It is natural to ask whether a system with t parameters has corresponding convexity properties to those of §2 when the parameters are allowed to vary simultaneously.

Let T be any simple t-vector, then, with K and $\mathscr{H}$ as above, we define $S(T, K, \mathscr{H})$ (the *shadow of K on $\mathscr{H}$ in the direction T*) *to be*

$$\mathscr{H} \cap Z(K, T)$$

where $Z(K, T)$ is the cylinder $\{x + y \mid x \in K, y \parallel T\}$ containing K and with t-dimensional generators parallel to T. Then

(19) $K(u_1, \cdots, u_t) = S(T, K, \mathscr{H})$ *where*

$$T = (a_1 + u_1) \wedge (a_2 + u_2) \wedge \cdots \wedge (a_t + u_t).$$

To see this, we notice that

$$\left(x - \sum_{i=1}^{t} \zeta_i u_i\right) + \sum_{i=1}^{t} \zeta_i (a_i + u_i) = x + \sum_{i=1}^{t} \zeta_i a_i,$$

so that the line joining $x - \sum \zeta_i u_i$ to $x + \sum \zeta_i a_i$ is linearly dependent on $a_1 + u_1, \cdots, a_t + u_t$ and so is parallel to T. Consequently $x - \sum \zeta_i u_i$ is the unique point which a t-dimensional subspace parallel to T, through the point $x + \sum \zeta_i a_i \in K$ meets $\mathscr{H}$. From this, (19) follows immediately.

We can show that

(20) *For the s-system defined by* (18),

$$v_n(K(u_1, \cdots, u_t)) = \hat{P}(K, T^{\perp})$$

where $T = (a_1 + u_1) \wedge \cdots \wedge (a_t + u_t)$ *and* $T^{\perp}$ *is the simple n-vector normal to* T *with* $|T^{\perp}| = |T|$.

(In terms of components,

$$T^{\perp}_{i_1 i_2 \ldots i_n} = T_{i_{n+1} i_{n+2} \ldots i_{n+t}}$$

if $(i_1 \cdots i_{n+t})$ is an even permutation of $(1, 2, \cdots, n + t)$.)

Let $T_0 = a_1 \wedge \cdots \wedge a_t$. Since $K(u_1, \cdots, u_t)$ and $P(K, \mathscr{T}^{\perp})$ (where $\mathscr{T}^{\perp}$ is the n-dimensional subspace of E^{n+t} through the origin determined by the vector $T^{\perp}$) are cross-sections of the same cylinder, the latter being the normal cross-section, we deduce

$$\frac{v_n(P(K, \mathscr{T}^{\perp}))}{v_n(K(u_1, \cdots, u_t))} = \frac{|T \cdot T_0|}{|T|\,|T_0|}.$$

However $|T_0| = 1$, $|T \cdot T_0| = 1$ from the way in which T and T_0 were defined, and $|T| = |T^{\perp}|$. Hence

$$\begin{aligned} v_t(K(u_1, \cdots, u_t)) &= |T^{\perp}|\, P(K, \mathscr{T}^{\perp}) \\ &= \hat{P}(K, T^{\perp}) \end{aligned}$$

by definition. Thus (20) is proved.

Relation (20) enables us to deduce immediately the properties of the system $\{K(u_1, \cdots, u_t)\}$ from the properties of $\hat{P}(K, R)$ given in [4] and [7]. For example, we see that if $n \geqq 2$, $v_n(K(u_1, \cdots, u_t))$ is not in general, a convex function of $T^{\perp}$ for $t > 1$, but is, in certain special cases. For example it is so if K is a simplex of at most $n + 2$ dimensions [7, p. 307] or if K is a vector sum of line segments [4, p. 20]. The fact that $v_n(K(u_1, \cdots, u_t))$ is, for all K, a convex function of each parameter u_i separately corresponds to the fact that $\hat{P}(K, T^{\perp})$ is a weakly convex function [4, p. 34], i.e. is convex on the generators of G_n^{n+t}.

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